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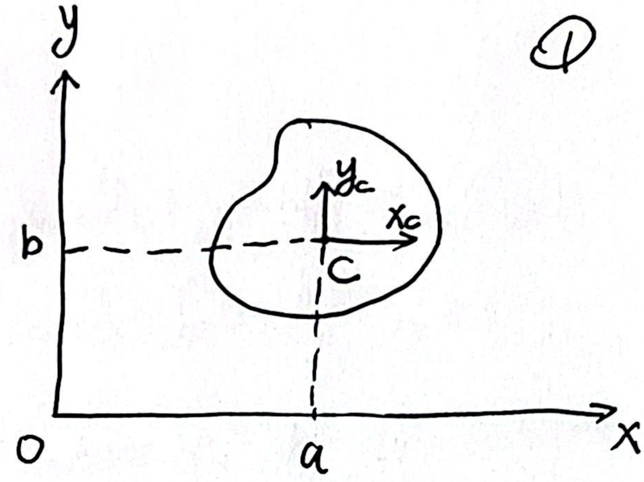
Twist Centre C



Relation between xoy and $x_c C y_c$:

$$\begin{cases} x_c = x - a \\ y_c = y - b \end{cases}$$

Shear stresses defined in xoy:

$$\tau_{xz} = G\alpha\left(\frac{\partial\psi}{\partial x} - y\right)$$

$$\tau_{yz} = G\alpha\left(\frac{\partial\psi}{\partial y} + x\right)$$

Here ψ is defined in xoy, too

Shear forces:

$$Q_x = \int_A \tau_{xz}\, dA = \int_A G\alpha\left(\frac{\partial\psi}{\partial x} - y\right) dA$$

$$= G\alpha\left(\int_A \frac{\partial\psi}{\partial x} dA - \int_A y\, dA\right)$$

$$= G\alpha\left(\psi_x - S_x\right)$$

$$Q_y = \int_A \tau_{yz}\, dA = \int_A G\alpha\left(\frac{\partial\psi}{\partial y} + x\right) dA$$

$$= G\alpha\left(\int_A \frac{\partial\psi}{\partial y} dA + \int_A x\, dA\right)$$

$$= G\alpha\left(-\psi_y + S_y\right)$$

Transform S_x, S_y into S_x^c, S_y^c.

$$S_x = \int_A y\, dA = \int_A (y_c + b)\, dA = \int_A y_c\, dA + \int_A b\, dA = S_x^c + bA$$

$$S_y = \int_A x\, dA = \int_A (x_c + a)\, dA = \int_A x_c\, dA + \int_A a\, dA = S_y^c + aA$$

Transform ψ_x, ψ_y into ψ_x^c, ψ_y^c where ψ_x^c, ψ_y^c are defined in $x_c C y_c$.

There are multiple ways to establish the relation, here we use shear stresses as the values must be the same regardless of coordinate system.

Using the $x_c C y_c$ system:

$$\tau_{xz}^c = G\alpha\left(\frac{\partial \psi^c}{\partial x_c} - y_c\right)$$

$$\tau_{yz}^c = G\alpha\left(\frac{\partial \psi^c}{\partial y_c} + x_c\right)$$

We have

$$\begin{cases} \tau_{xz}^c = \tau_{xz} \\ \tau_{yz}^c = \tau_{yz} \end{cases} \Rightarrow \begin{cases} G\alpha\left(\frac{\partial \psi^c}{\partial x_c} - y_c\right) = G\alpha\left(\frac{\partial \psi}{\partial x} - y\right) \\ G\alpha\left(\frac{\partial \psi^c}{\partial y_c} + x_c\right) = G\alpha\left(\frac{\partial \psi}{\partial y} + x\right) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial \psi^c}{\partial x_c} - y_c = \frac{\partial \psi}{\partial x} - y \\ \frac{\partial \psi^c}{\partial y_c} + x_c = \frac{\partial \psi}{\partial y} + x \end{cases}$$

Use $y_c = y - b$, $x_c = x - a$, we have

$$\begin{cases} \frac{\partial \psi^c}{\partial x_c} = \frac{\partial \psi}{\partial x} - b \\ \frac{\partial \psi^c}{\partial y_c} = \frac{\partial \psi}{\partial y} + a \end{cases} \xrightarrow[\text{integrate}]{\int_A dA} \begin{cases} \psi_x^c = \psi_x - bA \\ \psi_y^c = \psi_y - aA \end{cases}$$

Note: $dA = dx\,dy = d(x_c + a)\, d(y_c + b) = dx_c\, dy_c$

Using the relations above, we have

$$Q_x = G\alpha(\psi_x - S_x^c - bA) = G\alpha(\psi_x^c - S_x^c)$$

$$Q_y = G\alpha(-\psi_y + S_y^c + aA) = G\alpha(-\psi_y^c + S_y^c)$$

Shear forces should vanish, thus $Q_x = Q_y = 0$.

$$\begin{cases} G\alpha(\psi_x^c - S_x^c) = 0 \\ G\alpha(-\psi_y^c + S_y^c) = 0 \end{cases} \Rightarrow \begin{cases} \psi_x^c = S_x^c \\ \psi_y^c = S_y^c \end{cases}$$

Torque:

$$T = \int_A (\tau_{yz} x - \tau_{xz} y)\, dA$$

$$= \int_A \left(G\alpha\left(\frac{\partial \psi^c}{\partial y_c} + x_c\right)(x_c + a) - G\alpha\left(\frac{\partial \psi^c}{\partial x_c} - y_c\right)(y_c + b)\right) dA$$

$$= G\alpha \int_A \left(\left(x_c \frac{\partial \psi^c}{\partial y_c} + a\frac{\partial \psi^c}{\partial y_c} + x_c^2 + a x_c\right) - \left(y_c \frac{\partial \psi^c}{\partial x_c} + b\frac{\partial \psi^c}{\partial x_c} - y_c^2 - b y_c\right)\right) dA$$

$$= G\alpha \int_A \left(x_c \frac{\partial \psi^c}{\partial y_c} - y_c \frac{\partial \psi^c}{\partial x_c} + a\frac{\partial \psi^c}{\partial y_c} - b\frac{\partial \psi^c}{\partial x_c} + x_c^2 + y_c^2 + a x_c + b y_c\right) dA$$

$$= G\alpha\left(\left(\int_A \left(x_c \frac{\partial \psi^c}{\partial y_c} - y_c \frac{\partial \psi^c}{\partial x_c}\right) dA\right) - (a\psi_y^c + b\psi_x^c) + J_p^c + aS_y^c + bS_x^c\right)$$

Here, $\int_A \left(x_c \frac{\partial \psi^c}{\partial y_c} - y_c \frac{\partial \psi^c}{\partial x_c}\right) dA = -J_w^c$, $\int_A (x_c^2 + y_c^2)\, dA = J_p^c$,

both defined with c being the origin.

Proof for J_w^c is attached in 5

Then, with the definition,

$$T = G\alpha(-J_w^c + J_p^c - a\psi_y^c - b\psi_x^c + aS_y^c + bS_x^c)$$

$$= G\alpha(J_p^c - J_w^c + a(-\psi_y^c + S_y^c) + b(-\psi_x^c + S_x^c))$$

Because T is independent of a and b:

$$\begin{cases} -\psi_y^c + S_y^c = 0 \\ -\psi_x^c + S_x^c = 0 \end{cases} \Rightarrow \begin{cases} \psi_x^c = S_x^c \\ \psi_y^c = S_y^c \end{cases}$$

This is the same result from setting $Q_x = Q_y = 0$.

Therefore, we cannot combine the equations to conclude

$S_x^c = S_y^c = 0$ and $\psi_x = \psi_y = 0$.

Appendix: Proof $\int_A \left(x_c \frac{\partial \psi^c}{\partial y_c} - y_c \frac{\partial \psi^c}{\partial x_c}\right) dA = -J_w^c$

The Green's theorem states:

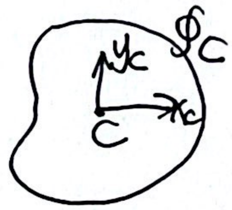
$$\oint_C (L\,dx_c + M\,dy_c) = \int_A \left(\frac{\partial M}{\partial x_c} - \frac{\partial L}{\partial y_c}\right) dA$$

Let $L = -x_c\psi^c$, $M = -y_c\psi^c$

$$\int_A \left(x_c \frac{\partial \psi^c}{\partial y_c} - y_c \frac{\partial \psi^c}{\partial x_c}\right) dA = -\oint_C (x_c\psi^c dx_c + y_c\psi^c dy_c)$$

$$= -\oint_C \left(\tfrac{1}{2}\psi^c dx_c^2 + \tfrac{1}{2}\psi^c dy_c^2\right)$$

$$= -\oint_C \tfrac{1}{2}\psi^c d(x_c^2 + y_c^2)$$



For BC, $\frac{\partial \psi^c}{\partial x_c} dy_c - \frac{\partial \psi^c}{\partial y_c} dx_c = \frac{1}{2} d(x_c^2 + y_c^2)$

So, the expression becomes $-\oint_C \psi^c \left(\frac{\partial \psi^c}{\partial x_c} dy_c - \frac{\partial \psi^c}{\partial y_c} dx_c\right)$

Use Green's theorem again by letting $M = \psi^c \frac{\partial \psi^c}{\partial x_c}$, $L = -\psi^c \frac{\partial \psi^c}{\partial y_c}$

$$-\oint \psi^c \left(\frac{\partial \psi^c}{\partial x_c} dy_c - \frac{\partial \psi^c}{\partial y_c} dx_c\right)$$

$$= -\int_A \left(\frac{\partial\left(\psi^c \frac{\partial \psi^c}{\partial x_c}\right)}{\partial x_c} - \frac{\partial\left(-\psi^c \frac{\partial \psi^c}{\partial y_c}\right)}{\partial y_c}\right) dA$$

$$= -\int_A \left(\frac{\partial \psi^c}{\partial x_c} \cdot \frac{\partial \psi^c}{\partial x_c} + \psi^c \frac{\partial^2 \psi^c}{\partial x_c^2} + \frac{\partial \psi^c}{\partial y_c} \cdot \frac{\partial \psi^c}{\partial y_c} + \psi^c \frac{\partial^2 \psi^c}{\partial y_c^2}\right) dA$$

$$= -\int_A \left(\left(\frac{\partial \psi^c}{\partial x_c}\right)^2 + \left(\frac{\partial \psi^c}{\partial y_c}\right)^2 + \psi^c \underline{\left(\frac{\partial^2 \psi^c}{\partial x_c^2} + \frac{\partial^2 \psi^c}{\partial y_c^2}\right)}\right) dA$$

GDE: $\frac{\partial^2 \psi^c}{\partial x_c^2} + \frac{\partial^2 \psi^c}{\partial y_c^2} = 0$

$$= -\int_A \left(\frac{\partial \psi^c}{\partial x_c}\right)^2 + \left(\frac{\partial \psi^c}{\partial y_c}\right)^2 dA = -J_w^c$$