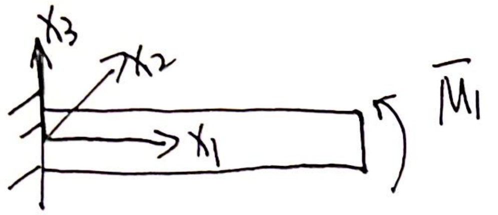


Apply torsion $\bar{M}_1$

Equilibrium: $\dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} = 0$ for pure torsion

σ_{12}, σ_{13}: shear stress

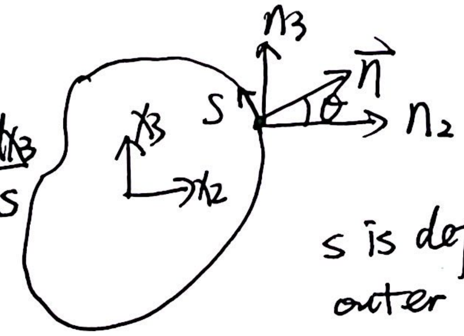
BC: traction free on outer surface

$$\vec{t} = \sigma \cdot \vec{n} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Arbitrary surface:

$n_1 = 0$ $\quad n_2 = \cos\theta = \dfrac{dx_3}{ds}$

$n_3 = -\sin\theta = -\dfrac{dx_2}{ds}$



s is defined along the outer surface

Based on resultant:

$$\bar{M}_1 = \iint \sigma_{13} x_2 - \sigma_{12} x_3 \, dA$$

$$= \iint -\frac{\partial \phi}{\partial x_2} x_2 - \frac{\partial \phi}{\partial x_3} x_3 \, dA$$

$$= -\iint \frac{\partial \phi}{\partial x_2} x_2 + \frac{\partial \phi}{\partial x_3} x_3 \, dA$$

→ Here $\sigma_{12} = \dfrac{\partial \phi}{\partial x_3}$, $\sigma_{13} = -\dfrac{\partial \phi}{\partial x_2}$

$\phi(x_2, x_3)$ is the stress function

Apply Green's theorem:

$$\iint \left(\frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} \right) dA = \oint_c (f_3 \, dx_2 + f_2 \, dx_3)$$

Let $f_2 = \phi x_2$, $f_3 = -\phi x_3$

$$\frac{\partial f_2}{\partial x_2} = \frac{\partial \phi}{\partial x_2} x_2 + \frac{\partial x_2}{\partial x_2} \phi = \frac{\partial \phi}{\partial x_2} x_2 + \phi$$

$$\frac{\partial f_3}{\partial x_3} = -\left(\frac{\partial \phi}{\partial x_3} x_3 + \frac{\partial x_3}{\partial x_3} \phi\right) = -\frac{\partial \phi}{\partial x_3} x_3 - \phi$$

$$\frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} = \frac{\partial \phi}{\partial x_2} x_2 + \phi + \frac{\partial \phi}{\partial x_3} x_3 + \phi = \frac{\partial \phi}{\partial x_2} x_2 + \frac{\partial \phi}{\partial x_3} + 2\phi$$

$$f_3 \, dx_2 + f_2 \, dx_3 = f_3 (-n_3 ds) + f_2 n_2 ds \rightarrow \begin{cases} n_2 = \frac{dx_3}{ds} \\ n_3 = -\frac{dx_2}{ds} \end{cases}$$

$$= +\phi x_3 n_3 ds + \phi x_2 n_2 ds$$

$$= \phi (x_2 n_2 + x_3 n_3) ds$$

Equation for Green's theorem becomes:

$$\iint \frac{\partial \phi}{\partial x_2} x_2 + \frac{\partial \phi}{\partial x_3} + 2\phi \, dA = \oint_c \phi (x_2 n_2 + x_3 n_3) ds$$

Therefore,

$$\overline{M}_1 = -\iint \frac{\partial \phi}{\partial x_2} x_2 + \frac{\partial \phi}{\partial x_3} x_3 \, dA$$

$$= -\iint \frac{\partial \phi}{\partial x_2} x_2 + \frac{\partial \phi}{\partial x_3} x_3 + 2\phi \, dA + \iint 2\phi \, dA$$

$$= -\oint_c \phi (x_2 n_2 + x_3 n_3) ds + \iint 2\phi \, dA$$

Assume torsion is not coupled with other beam modes, $\bar{M}_1 = GJ\kappa_1$ where G is shear modulus, J is torsional constant, κ_1 is twist rate.

Then, $\bar{M}_1 = \iint 2\phi\, dA - \oint_c \phi(x_2 n_2 + x_3 n_3)\, ds = GJ\kappa_1$

For cross-section bound by a single curve, ϕ can be chosen as $\phi = 0$ at the outer boundary. Therefore,

$\bar{M}_1 = \iint 2\phi\, dA = GJ\kappa_1$

stress strain relationship

$\sigma_{12} = G(2\varepsilon_{12}) \qquad \sigma_{13} = G(2\varepsilon_{13})$

Define warping function:

$2\varepsilon_{12} = \left(\frac{\partial\psi}{\partial x_2} - x_3\right)\kappa_1 \qquad 2\varepsilon_{13} = \left(\frac{\partial\psi}{\partial x_3} + x_2\right)\kappa_1$

Then

$$\bar{M}_1 = \iint \sigma_{13}x_2 - \sigma_{12}x_3\, dA$$

$$= \iint 2G\varepsilon_{13}x_2 - 2G\varepsilon_{12}x_3\, dA$$

$$= \iint G\left(\frac{\partial\psi}{\partial x_3} + x_2\right)\kappa_1 x_2 - G\left(\frac{\partial\psi}{\partial x_2} - x_3\right)\kappa_1 x_3\, dA$$

$$= G\kappa_1 \iint \left(x_2^2 + x_3^2 + x_2\frac{\partial\psi}{\partial x_3} - x_3\frac{\partial\psi}{\partial x_2}\right) dA = GJ\kappa_1$$

Using displacement strain relation:

$$\begin{cases} 2\varepsilon_{12} = \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} = \left(\dfrac{\partial \psi}{\partial x_2} - x_3\right)\kappa_1 & (1) \\ 2\varepsilon_{13} = \dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1} = \left(\dfrac{\partial \psi}{\partial x_3} + x_2\right)\kappa_1 & (2) \\ 2\varepsilon_{23} = \dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2} = 0 & (3) \end{cases}$$

$$\begin{cases} \varepsilon_{11} = \dfrac{\partial u_1}{\partial x_1} = 0 \Rightarrow u_1 = f(x_2, x_3) \text{ not a function of } x_1 \\ \varepsilon_{22} = \dfrac{\partial u_2}{\partial x_2} = 0 \Rightarrow u_2 = f(x_1, x_3) \\ \varepsilon_{33} = \dfrac{\partial u_3}{\partial x_3} = 0 \Rightarrow u_3 = f(x_1, x_2) \end{cases}$$

From (1):

$$\frac{\partial u_1(x_2, x_3)}{\partial x_2} + \frac{\partial u_2(x_1, x_3)}{\partial x_1} = \left(\frac{\partial \psi}{\partial x_2} - x_3\right)\kappa_1$$

$$u_2 = -x_1 \frac{\partial u_1}{\partial x_2} + \left(\frac{\partial \psi}{\partial x_2} - x_3\right)\kappa_1 x_1 + \underbrace{c_2(x_3)}_{\text{function of } x_3}$$

From (2):

$$\frac{\partial u_1(x_2, x_3)}{\partial x_3} + \frac{\partial u_3(x_1, x_2)}{\partial x_1} = \left(\frac{\partial \psi}{\partial x_2} - x_3\right)\kappa_1$$

$$u_3 = -x_1 \frac{\partial u_1}{\partial x_3} + \left(\frac{\partial \psi}{\partial x_3} + x_2\right)\kappa_1 x_1 + c_1(x_2)$$

From (3):

$$\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 2\frac{\partial^2 \psi}{\partial x_2 \partial x_3}\kappa_1 x_1 - 2\frac{\partial^2 u_1}{\partial x_2 \partial x_3} x_1 + \frac{\partial c_2(x_3)}{\partial x_3} + \frac{\partial c_1(x_2)}{\partial x_2} = 0$$

From the coefficient of x_1:

$$2\frac{\partial^2\psi}{\partial x_2\partial x_3}x_1 - 2\frac{\partial^2 u_1}{\partial x_2\partial x_3} = 0$$

$$\frac{\partial^2 u_1}{\partial x_2\partial x_3} = \frac{\partial^2\psi}{\partial x_2\partial x_3}x_1 \Rightarrow \frac{\partial u_1}{\partial x_2} = \frac{\partial\psi}{\partial x_2}x_1 + C_6(x_2) \Rightarrow u_1 = \psi x_1 + \int C_6(x_2)\,dx_2 + C_7(x_3)$$

From the functions of x_2, x_3:

$$\frac{\partial C_2(x_3)}{\partial x_3} = -C_3 \Rightarrow C_2(x_3) = -C_3x_3 + C_5$$

$$\frac{\partial C_1(x_2)}{\partial x_2} = C_3 \Rightarrow C_1(x_2) = C_3x_2 + C_4$$

Therefore, $\frac{\partial C_2(x_3)}{\partial x_3} + \frac{\partial C_1(x_2)}{\partial x_2} = -C_3 + C_3 = 0$

Substitute $C_1(x_2)$, $C_2(x_3)$:

$$u_2 = -x_1\frac{\partial u_1}{\partial x_2} + \left(\frac{\partial\psi}{\partial x_2} - x_3\right)x_1x_1 - C_3x_3 + C_5$$

$$u_3 = -x_1\frac{\partial u_1}{\partial x_3} + \left(\frac{\partial\psi}{\partial x_3} + x_2\right)x_1x_1 + C_3x_2 + C_4$$

Substitite $u_1 = \psi x_1 + \int C_6(x_2)dx_2 + C_7(x_3)$ into u_2

$$u_2 = -x_1\left(\frac{\partial\psi}{\partial x_2}x_1 + C_6(x_2)\right) + \left(\frac{\partial\psi}{\partial x_2} - x_3\right)x_1x_1 - C_3x_3 + C_5$$

$$= (-C_6(x_2) - x_3x_1)x_1 - C_3x_3 + C_5$$

However, u_2 is a function of x_1, x_3 only. Therefore, $C_6(x_2) = C_6$

$$u_2 = -x_1x_1x_3 - C_3x_3 + C_5 - C_6x_1$$

Substitute $u_1 = \psi x_1 + \int C_6 dx_2 + G(x_3) = \psi x_1 + C_6 x_2 + C_9 + C_7(x_3)$ ⑥

into u_3:

$$u_3 = -x_1\left(\frac{\partial\psi}{\partial x_3}x_1 + \frac{\partial C_7(x_3)}{\partial x_3}\right) + \left(\frac{\partial\psi}{\partial x_3} + x_2\right)\chi_1 x_1 + C_3 x_2 + C_4$$

$$= \left(-\frac{\partial C_7(x_3)}{\partial x_3} + x_2 \chi_1\right)x_1 + C_3 x_2 + C_4$$

Similarly, u_3 is only a function of x_1, x_2, therefore,

$$\frac{\partial C_7(x_3)}{\partial x_3} = C_8 \Rightarrow C_7(x_3) = C_8 x_3 + C_{10}$$

$$u_3 = (-C_8 + x_2\chi_1)x_1 + C_3 x_2 + C_4 = \chi_1 x_1 x_2 - C_8 x_1 + C_3 x_2 + C_4$$

Substitute $C_7(x_3)$ into u_1:

$$u_1 = \psi x_1 + C_6 x_2 + C_9 + C_8 x_3 + C_{10} \equiv \psi x_1 + C_6 x_2 + C_8 x_3 + C_9$$

Note here C_{10} is absorbed into C_9

The displacement field is

$$\begin{cases} u_1 = \psi x_1 + C_6 x_2 + C_8 x_3 + C_9 \\ u_2 = -\chi_1 x_1 x_3 - C_3 x_3 + C_5 - C_6 x_1 \\ u_3 = \chi_1 x_1 x_2 - C_8 x_1 + C_3 x_2 + C_4 \end{cases}$$

where $C_3, C_4, C_5, C_6, C_8, C_9$ are constants that need to be determined using 6 BCs.

It is clear that the displacement field is dependant on the BCs. Therefore, twist center, if defined as the stationary point: $u_2 = u_3 = 0$ is sensitive to the BCs applied.

Case 1: cantilever beam

Apply weak BCs so the average of displacement and rotation are zero:

$$\begin{cases} \int u_1 dA = \int u_2 dA = \int u_3 dA = 0 \\ \int \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} dA = \int \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} dA = \int \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} dA = 0 \end{cases}$$

Solve for 6 constants

$$\begin{cases} C_3 = C_4 = C_5 = 0 \\ C_6 = -\frac{\kappa_1}{2}\left(\frac{\int \psi_{,2} dA}{A} + \frac{\int x_3 dA}{A}\right) \\ C_8 = -\frac{\kappa_1}{2}\left(\frac{\int \psi_{,3} dA}{A} - \frac{\int x_2 dA}{A}\right) \\ C_9 = -\frac{\int \psi dA}{A} x_1 + \frac{\kappa_1}{2}\left(\frac{\int \psi_{,2} dA}{A} \cdot \frac{\int x_2 dA}{A} + \frac{\int \psi_{,3} dA}{A} \cdot \frac{\int x_3 dA}{A}\right) \end{cases}$$

Define $\langle \psi \rangle = \frac{\int \psi dA}{A}$, $x_{t\alpha} = \frac{\int x_\alpha dA}{A}$ for convenience.

Substitute back into u_2, u_3

$$u_2 = -\kappa_1 x_1 x_3 + \frac{\kappa_1}{2}(\langle \psi_{,2} \rangle + x_{t3}) x_1$$

$$u_3 = \kappa_1 x_1 x_2 + \frac{\kappa_1}{2}(\langle \psi_{,3} \rangle - x_{t2}) x_1$$

Let $u_2 = u_3 = 0$

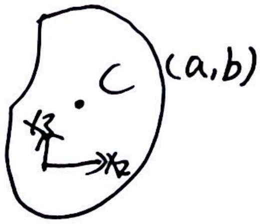
$$\begin{cases} u_2 = x_1\left(-\chi_1 x_3 + \frac{\chi_1}{2}\left(\langle \psi_{,2}\rangle + x_{t3}\right)\right) = 0 \\ u_3 = x_1\left(\chi_1 x_2 + \frac{\chi_1}{2}\left(\langle \psi_{,3}\rangle - x_{t2}\right)\right) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_3 = \frac{1}{2}\left(\langle \psi_{,2}\rangle + x_{t3}\right) \\ x_2 = -\frac{1}{2}\left(\langle \psi_{,3}\rangle - x_{t2}\right) \end{cases}$$

Here x_{t2}, x_{t3} are the location of the centroid.

Case 2: Fix displacement and rotation at one point inside the cross-section

Fix point C (a,b) where a, b are x_2, x_3 coordinates of point C.



BCs:
$$\begin{cases} u_1(0,a,b) = u_2(0,a,b) = u_3(0,a,b) = 0 \\ \dfrac{\partial u_1}{\partial x_2} - \dfrac{\partial u_2}{\partial x_1} = \dfrac{\partial u_2}{\partial x_3} - \dfrac{\partial u_3}{\partial x_2} = \dfrac{\partial u_1}{\partial x_3} - \dfrac{\partial u_3}{\partial x_1} = 0 \\ \text{at } (0, a, b) \end{cases}$$

Solve for 6 constants:

$$\begin{cases} C_3 = C_4 = C_5 = 0 \\ C_6 = -\frac{\chi_1}{2}\left(b + \psi_{,2}(a,b)\right) \\ C_8 = \frac{\chi_1}{2}\left(a - \psi_{,3}(a,b)\right) \\ C_9 = -\chi_1 \psi(a,b) + \frac{\chi_1}{2}\left(a\psi_{,2}(a,b) + b\psi_{,3}(a,b)\right) \end{cases}$$

$$u_2 = \frac{x_1 x_1}{2}(b - 2x_3 + \psi_{,2}(a,b))$$

$$u_3 = \frac{x_1 x_1}{2}(-a + 2x_2 + \psi_{,3}(a,b))$$

Let $u_2 = u_3 = 0$

$$\begin{cases} x_2 = \frac{1}{2}(a - \psi_{,3}(a,b)) \\ x_3 = \frac{1}{2}(b + \psi_{,2}(a,b)) \end{cases}$$

This means the twist center is a constant depending on ψ and the location of the fixedpoint c.